— Exercises Week 12: End Projects —

For the most part, the following problems are <u>not</u> exercises: They require some ideas. The exercise consists of reading and understanding their proofs. After each topic I write in gray some comments about it and I give some References where you can read about them.

Topic 1) Other models for the hyperbolic plane

N.B. you can only choose this topic if you didn't know what a Riemannian metric is before starting this course

Study the hyperboloid and the projective (a.k.a Klein) models for the hyperbolic plane.

(i.e. understand why they are isometric to the Poincaré disk and Upper Half-Plane model. Describe geodesics & isometries)

Reference [Martelli: Section 2.1]

Topic 2) The Dehn Lickorish Theorem

Prove that $MCG(\Sigma)$ is generated by finitely many Dehn Twists

References: [Martelli: Section 6.5.4] [Farb-Margalit: Chapter 4]

Topic 3) Convexity of length functions:

Let $\gamma \subset \Sigma$ be a simple closed curve, and $\phi : \Sigma \to (\Sigma, d)$ a hyperbolic marking. Choose a hexagon decomposition $(\mathcal{P}, \mathcal{D})$ such that $\gamma = \gamma_1$ is the first curve of $\mathcal{P}$, and let

$$(\ell_1, \dots, \ell_{3g-3}, \tau_1, \dots, \tau_{3g-3}) \in \mathbb{R}^{3g-3}_{>0} \times \mathbb{R}^{3g-3}$$

be the Fenchel-Nielsen coordinates of $[\phi : \Sigma \to (\Sigma, d)]$

For every $s \in \mathbb{R}$, let $[\phi_s : \Sigma \to (\Sigma, d_s)]$ be the element of $\mathrm{Teich}(\Sigma)$ having coordinates $(\ell_1, \dots, \ell_{3g-3}, \boxed{\tau_1 + s}, \tau_2, \dots, \tau_{3g-3})$

(i.e. ϕ_s is obtained from ϕ changing by s the torsion parameter associated with γ)

Let $\alpha \subset \Sigma$ any curve such that $i([\alpha],[\gamma]) > 0$.

Show that the function
$$\begin{aligned} \mathbb{R} &\longrightarrow \mathbb{R}_{>0} \\ s &\longmapsto \left(\begin{array}{l} \text{length of the unique geodesic} \\ \text{in } (\Sigma, d_s) \text{ homotopic to } \phi(\alpha) \end{array} \right) \end{aligned}$$
is strictly convex.

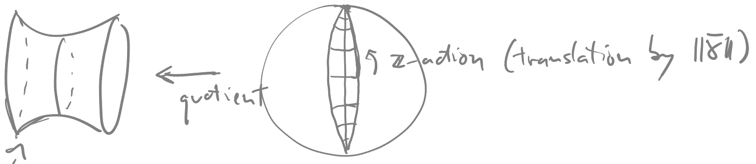
References: [Farb-Margalit, Section 10.7.3] or [Martelli, Section 7.2]

Recall: We used this fact to show that there is an embedding
$$\mathrm{Teich}(\Sigma) \hookrightarrow \mathbb{R}^{9g-9}_{>0}$$

It is interesting to note (exercise) that the element $[\phi_s] \in \mathrm{Teich}(\Sigma)$ depends only on $[\gamma]$, $[\phi]$ and s. That is, if we had chosen a different hexagon decomposition to define the Fenchel-Nielsen coordinates, we would have obtained the same element $[\phi_s]$.

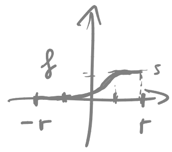
In other words, once we choose a homotopy class $[\gamma]$ of an (oriented) simple closed curve, we have a natural map
$$\begin{aligned} \mathbb{R} \times \mathrm{Teich}(\Sigma) &\longrightarrow \mathrm{Teich}(\Sigma) \\ (s, [\phi]) &\longmapsto [\phi_s] \end{aligned}$$
This map is a group action of $\mathbb{R}$ on $\mathrm{Teich}(\Sigma)$ which is called the earthquake action.

A more direct way of defining $[\phi_s]$ is as follows: let $\bar\gamma$ be the geodesic representative of $\phi(\gamma)$ in (Σ,d). There is $r>0$ small enough so that the r-neighbourhood of $\bar\gamma$ in (Σ,d) is isometric to a hyperbolic cylinder:



$S^1\times[-r,r]$ with a hyperbolic metric (the boundary is not geodesic)

Choose a smooth function $f_s:[-r,r]\longrightarrow\mathbb{R}$ that starts at 0 and ends at s



and let
$$\boxed{\begin{aligned} F:\ S^1\times[-r,r] &\longrightarrow S^1\times[-r,r]\\ (\theta,t)&\longmapsto(\theta+f(t),t)\end{aligned}}.$$

The push-forward F_*d induces a new hyperbolic metric on the cylinder, which glues nicely to give a hyperbolic metric d_s on Σ

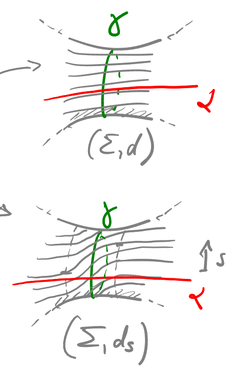
(Notice that F does not extend to a homeomorphism $\Sigma\to\Sigma$ inducing d_s. Such an extension only exists when s is a multiple of 2π)

Letting $\phi_s:=\phi$ (as set-theoretic functions), we defined a marking $\phi_s:\Sigma\longrightarrow(\Sigma,d_s)$. It is easy to check that the choice of r and f does not influence the equivalence class $[\phi_s:\Sigma\to(\Sigma,d_s)]$

(different choices will yield different metrics d_s', but there will be an isometry $G:(\Sigma,d_s)\to(\Sigma,d_s')$ that is homotopic to id_Σ)

This shows that $[\phi_s]$ is a well-defined element on $\mathrm{Teich}(\Sigma)$, depending only on $[\gamma]$, $[\phi]$, $s\in\mathbb{R}$.

$\boxed{\text{Rmk}}$ it follows from the definition of ϕ_s that if these are geodesics in (Σ,d), then these are geodesics in (Σ,d_s). In particular, the curve α is further away from being geodesic in (Σ,d_s). This is why the length of α varies as the torsion parameter associated with γ varies



It is also possible to generalize the definition of earthquakes to geodesic laminations. This is useful to solve problems such as $\underbrace{\text{Nielsen's Realization Problem}}$ [Kerckhoff]

every finite sgp of $MCG(\Sigma)$ can be realized as a finite subgroup of $\mathrm{Homeo}(\Sigma)$

Topic 4) Grötzsch Problem

Study the definition of quasi-conformality and show that if a quasi conformal homeomorphism $f : [0,1] \times [0,1] \longrightarrow [0,1] \times [0,r]$ sending vertices to vertices is K-quasi-conformal then $K \geq r$.
Furthermore, f is r-quasi-conformal i.f.f. it is affine

Reference : [Farb-Margalit : Sections 11.1 & 11.5]

Teichmüller generalized this result to the context of homeomorphisms among hyperbolic surfaces. This leads to the definition of the Teichmüller metric on $\mathrm{Teich}(\Sigma)$ and the description of the geodesics in it.

Topic 5) Holomorphic Quadratic differentials

Read and explain [Farb-Margalit : Section 11.3].

These are useful for proving Teichmüller Theorem on the existence and uniqueness of quasi-conformal maps realizing the minimal quasi-conformality constant.
They are also a convenient way for describing measured foliations

Topic 6) Nielsen-Thurston Classification for the torus

Show that a homeomorphism of the torus $F : \mathbb{T} \to \mathbb{T}$ must satisfy one of the following

- $[F]$ has finite order in $MCG(\mathbb{T})$ (elliptic)
- F fixes one simple closed curve up to homotopy (parabolic)
- F is Anosov (hyperbolic)

Up to homotopy:
there are two transverse foliations (with irrational slopes) that are fixed by F.
F stretches and dilates their transverse measures by a factor $\lambda > 1$.

Reference: [Martelli: Section 8.1.3] [Farb-Margalit: Section 13.1]

This fact was very well-known and presumably it served as inspiration to Thurston (and many others).

Topic 7) A Theorem of Smale

Let $F: \mathbb{D}^2 \longrightarrow \mathbb{D}^2$ be a C^∞ diffeomorphism that restricts to the identity on a neighbourhood of $\partial \mathbb{D}^2$. Show that there exists a smooth isotopy $\boxed{H: \mathbb{D}^2 \times [0,1] \longrightarrow \mathbb{D}}$ taking F to $id_{\mathbb{D}^2}$

C^∞ $\quad$ $H(-,t)$ is a diffeo $\forall t \in [0,1]$

References [Martelli, Section 6.4.2] [Smale: "Diffeomorphisms of the 2-sphere". Proc. Am. Math. Soc, '59]

Remember that in class we used Alexander's Trick to prove that any homeomorphism $\mathbb{D}^2 \to \mathbb{D}^2$ that fixes the boundary is isotopic to the identity, but that isotopy was very much not smooth.

This theorem allows us to prove that two diffeomorphisms $\Sigma \to \Sigma$ are homotopic iff they are smoothly isotopic (we stated as a fact and sort of proved in the exercises that homotopic homeomorphisms are (non-smoothly) isotopic).

Knowing that you can do things smoothly can be very convenient (e.g. if you want to talk about "transversality" and "differentials" as you isotope things around). These matters become crucial when going to higher dimensions, because the relation between smooth and topological structures become much more delicate.

It is also necessary when doing operations such as "glueing". For example, let $M_\phi := \Sigma \times [0,1] / (x,1) \sim (\phi(x), 0)$ be the mapping torus of $\phi: \Sigma \circlearrowleft$. If $\phi_0, \phi_1 : \Sigma \circlearrowleft$ are homotopic diffeos, let $H: \Sigma \times I \to X$ be a smooth isotopy between them: $H(-,0) = \phi_0$, $H(-,1) = \phi_1$.

We then have a diffeomorphism $F: \Sigma \times [0,1] \longrightarrow \Sigma \times [0,1]$ which passes to
$$(x,t) \longmapsto (H(x,t), t)$$
the quotient and gives rise to a diffeomorphism $M_{\phi_1} \longrightarrow M_{\phi_2}$: